Chapter 19

Quadratic equations

Contents:

Equations of the form $ax^2+bx+c=0$ where $a \neq 0$ are called **quadratic equations**.

For example, $2x^2 - 3x + 5 = 0$ and $x^2 + 3x = 0$ are quadratic equations

whereas $x^2 + 5 + \frac{1}{x} = 0$ is not.

Consider the equation $x^2 = 4x$.

One might be tempted to divide both sides by x to get $x = 4$. However, in doing this the obvious solution $x = 0$ is lost.

We therefore need another method of solution to ensure that no solutions are lost.

A THE NULL FACTOR LAW

The Null Factor law is an essential part of the process for solving quadratic equations.

THE NULL FACTOR LAW

When the product of two or more numbers is zero, then at least one of them must be zero.

So, if $ab = 0$ then $a = 0$ or $b = 0$.

For example:
- if $2xy = 0$ then $x = 0$ or $y = 0$
- if $x(x-2) = 0$ then $x = 0$ or $x - 2 = 0$
- if $xyz = 0$ then $x = 0$, $y = 0$, or $z = 0$.

Example 1 Self Tutor

Solve for x: **a** $5x(x+2) = 0$ **b** $(x+4)(x-1) = 0$

a $5x(x+2) = 0$

$\therefore$ $5x = 0$ or $x + 2 = 0$ {Null Factor law}

$\therefore$ $x = 0$ or $x = -2$ {solving linear equations}

So, $x = 0$ or -2

b $(x+4)(x-1) = 0$

$\therefore$ $x + 4 = 0$ or $x - 1 = 0$ {Null Factor law}

$\therefore$ $x = -4$ or $x = 1$ {solving linear equations}

So, $x = -4$ or 1

EXERCISE 19A

1 Explain what can be deduced from:

a $ac = 0$ **b** $bd = 0$ **c** $abc = 0$ **d** $3x = 0$

e $x(x-3) = 0$ **f** $x^2 = 0$ **g** $(x-5)y = 0$ **h** $x^2y = 0$

2 Solve for x:

a $2x(x-1)=0$ b $x(x+5)=0$ c $3x(x+2)=0$
d $(x-1)^2=0$ e $-x(x-4)=0$ f $-2x(x+3)=0$
g $x(2x+1)=0$ h $3x(4x-3)=0$

3 Solve for x:

a $(x-1)(x-5)=0$ b $(x+2)(x-4)=0$ c $(x+3)(x+7)=0$
d $(x+7)(x-11)=0$ e $2x(x-8)=0$ f $(x+12)(x-5)=0$
g $-3x(x+7)=0$ h $(2x+1)(x-3)=0$ i $(x+6)(3x-1)=0$
j $(2x+1)(x+6)=0$ k $x(3x+5)=0$ l $(x-31)(x+11)=0$
m $(x+4)(4x-1)=0$ n $-3x(x+3)=0$ o $(2-x)(3x+4)=0$

B EQUATIONS OF THE FORM $ax^2+bx=0$

To solve quadratic equations of the form $ax^2+bx=0$ where $a\neq 0$ we first of all take out x as a common factor. We can then use the *Null Factor law*.

Example 2 Self Tutor

Solve for x: $x^2=4x$

$x^2=4x$
$\therefore\ x^2-4x=0$ {subtracting $4x$ from both sides to make RHS $=0$}
$\therefore\ x(x-4)=0$ {factorising the LHS}
$\therefore\ x=0$ or $x-4=0$ {using the Null Factor law}
$\therefore\ x=0$ or $x=4$

$\frac{x^2}{x}=\frac{4x}{x}$ gives $x=4$ only. We need the Null Factor law to ensure the solution $x=0$ is not lost.

EXERCISE 19B

1 Solve for x:

a $x^2-x=0$ b $x^2-13x=0$ c $x^2+8x=0$
d $x^2+3x=0$ e $2x+x^2=0$ f $5x-x^2=0$
g $12x-x^2=0$ h $x^2+7x=0$ i $x^2-4x=0$
j $2x^2-7x=0$ k $3x^2-15x=0$ l $2x^2+8x=0$

2 Solve for x:

a $x^2=3x$ b $x^2=10x$ c $x^2+x=0$
d $x^2=-6x$ e $2x^2+7x=0$ f $5x=x^2$
g $8x=x^2$ h $3x^2=18x$ i $5x^2=6x$
j $7x+x^2=x$ k $x^2-4=2x-4$ l $2x^2+x=3x$

C SOLVING EQUATIONS USING THE 'DIFFERENCE OF SQUARES'

Equations like $x^2-9=0$ and $4x^2-1=0$ have the 'difference of squares' on the LHS of the equals sign.

We can use $a^2-b^2=(a+b)(a-b)$ to convert the difference into a product.

Example 3 Self Tutor

Solve for x: a $x^2-4=0$ b $x^2=9$

a
$x^2-4=0$
$\therefore\ (x+2)(x-2)=0$
$\therefore\ x+2=0$ or $x-2=0$
$\therefore\ x=-2$ or 2
So, $x=\pm 2$

b
$x^2=9$
$\therefore\ x^2-9=0$
$\therefore\ (x+3)(x-3)=0$
$\therefore\ x+3=0$ or $x-3=0$
$\therefore\ x=-3$ or 3
So, $x=\pm 3$

EXERCISE 19C

1 Solve for x:

a $x^2-16=0$ b $x^2-49=0$ c $x^2-1=0$
d $x^2=25$ e $x^2=144$ f $x^2=81$
g $2x^2-8=0$ h $3x^2-27=0$ i $5x^2-20=0$
j $-3x^2+12=0$ k $-2x^2+8=0$ l $x^2+4=0$

2 Solve for x:

a $2-8x^2=0$ b $1-9x^2=0$ c $4-25x^2=0$ d $4x^2-1=0$
e $4x^2-9=0$ f $9x^2-4=0$ g $3x^2+12=0$ h $16x^2-25=0$

D SOLVING EQUATIONS OF THE FORM $x^2 + bx + c = 0$

To solve $x^2 - 5x + 6 = 0$ we must first factorise $x^2 - 5x + 6$.

To do this we have to find two numbers with a **sum** of -5 and a **product** of 6.

The numbers required are -2 and -3, so $x^2 - 5x + 6 = (x-2)(x-3)$.

Example 4 Self Tutor

Solve for x: **a** $x^2 - 3x + 2 = 0$ **b** $x^2 = x + 12$ **c** $x^2 + 4 = 4x$

a $x^2 - 3x + 2 = 0$ {We need two numbers with sum -3 and product 2. These are -1 and -2.}
$\therefore (x-1)(x-2) = 0$
$\therefore x - 1 = 0$ or $x - 2 = 0$
$\therefore x = 1$ or 2

b $x^2 = x + 12$ {Make RHS $= 0$ by subtracting $x + 12$ from both sides}
$\therefore x^2 - x - 12 = 0$ {We need two numbers with sum -1 and product -12. These are -4 and $+3$.}
$\therefore (x-4)(x+3) = 0$
$\therefore x - 4 = 0$ or $x + 3 = 0$
$\therefore x = 4$ or -3

c $x^2 + 4 = 4x$ {Make RHS $= 0$ by subtracting $4x$ from both sides}
$\therefore x^2 - 4x + 4 = 0$ {We need two numbers with sum -4 and product $+4$. These are -2 and -2.}
$\therefore (x-2)(x-2) = 0$
$\therefore x = 2$

Sometimes each term in an equation contains a **constant common factor**. We can simplify these equations by dividing each term by the constant.

For example, consider $3x^2 + 21x + 30 = 0$.
If we divide each term by 3 then $x^2 + 7x + 10 = 0$.

EXERCISE 19D

1 Solve for x:

a $x^2 + 7x + 10 = 0$ **b** $x^2 + 6x + 8 = 0$ **c** $x^2 + 11x + 10 = 0$
d $x^2 - 8x + 12 = 0$ **e** $x^2 - 5x + 4 = 0$ **f** $x^2 - 11x + 24 = 0$
g $x^2 + 10x + 25 = 0$ **h** $x^2 - 3x - 18 = 0$ **i** $x^2 + 7x - 18 = 0$
j $x^2 - 22x + 121 = 0$ **k** $x^2 - 6x + 9 = 0$ **l** $x^2 - 5x - 6 = 0$
m $x^2 + 11x - 60 = 0$ **n** $x^2 + 18x - 63 = 0$ **o** $x^2 - 12x - 64 = 0$

2 Solve for x:

a $3x^2 + 21x + 30 = 0$ **b** $2x^2 + 4x - 30 = 0$ **c** $2x^2 - 24x + 72 = 0$
d $3x^2 - 21x + 36 = 0$ **e** $5x^2 - 5x - 210 = 0$ **f** $4x^2 + 32x + 48 = 0$

3 Solve for x:

a $x^2 - 14x = 15$ **b** $x^2 + 2 = 3x$ **c** $x^2 = 3x + 28$
d $x^2 = 20 + x$ **e** $8 = x^2 + 7x$ **f** $x^2 = 5x + 24$
g $2x^2 + 2x = 24$ **h** $x^2 = 4x + 45$ **i** $3x^2 = 30x - 48$
j $x^2 + 1 = 2x$ **k** $x^2 = 19x + 20$ **l** $5x^2 = 20(x + 8)$

4 Solve for x:

a $x(x+2) = 15$ **b** $x(x-2) = 5(x+12)$ **c** $2x - 6 = x(x-5)$
d $x^2 - 4 = x + 2$ **e** $2(x+5) = x^2 + 11$ **f** $5 - x^2 = 2x - 3$

INVESTIGATION SOLUTIONS OF A QUADRATIC EQUATION

How many solutions can a quadratic equation have?

This investigation should help to answer this question.

What to do:

1 Show that $(x-1)^2 = 4$, $(x-1)^2 = 0$ and $(x-1)^2 = -4$ can all be put into quadratic form $ax^2 + bx + c = 0$.

2 **a** Show that $(x-1)^2 = 4$ has two different solutions.
b Show that $(x-1)^2 = 0$ has one solution.
c Explain why $(x-1)^2 = -4$ has no solutions.

3 Use the graphing package to graph $y = (x-1)^2$.

GRAPHING PACKAGE

a Now graph $y = 4$.
The graphs meet when $(x-1)^2 = 4$.
Use this to solve the equation $(x-1)^2 = 4$.
b Erase the graph of $y = 4$ and replace it with $y = 0$.
Use this to solve the equation $(x-1)^2 = 0$.
c Erase the graph of $y = 0$ and replace it with $y = -4$.
Use this to solve the equation $(x-1)^2 = -4$.

E PROBLEM SOLVING WITH QUADRATIC EQUATIONS

Many problems when converted to algebraic form result in a **quadratic equation**.

We use **factorisation** and the **Null Factor law** to solve these equations.

PROBLEM SOLVING METHOD

- Carefully **read the question** until you **understand** it. A **rough sketch** may be useful.
- Decide on the **unknown** quantity. Label it with a variable such as x.
- Find an **equation** which connects x and the information you are given.
- **Solve** the equation using factorisation and the **Null Factor law**.
- **Check** that any solutions satisfy the original problem.
- Write your answer to the question in **sentence form**.

Example 5 Self Tutor

The sum of a number and its square is 30. Find the number.

Let the number be x.

$$\begin{aligned} \text{So, } \quad x + x^2 &= 30 && \{\text{the number plus its square is } 30\} \\ \therefore \quad x^2 + x &= 30 && \{\text{rearranging}\} \\ \therefore \quad x^2 + x - 30 &= 0 && \{\text{subtracting 30 from both sides}\} \\ \therefore \quad (x+6)(x-5) &= 0 && \{\text{factorising}\} \\ \therefore \quad x = -6 \quad &\text{or} \quad x = 5 \end{aligned}$$

$\therefore$ the numbers are -6 and 5.

Check: If $x = -6$, we have $-6 + (-6)^2 = -6 + 36 = 30$ ✓
If $x = 5$, we have $5 + 5^2 = 5 + 25 = 30$ ✓

Example 6 Self Tutor

Two numbers have a sum of 10 and the sum of their squares is 58. Find the numbers.

Let one of the numbers be x.

$$\begin{aligned} \therefore \quad &\text{the other number is } 10 - x. && \{\text{their sum is } 10\} \\ \text{So, } \quad &x^2 + (10-x)^2 = 58 && \{\text{insert brackets around } 10 - x\} \\ \therefore \quad &x^2 + 100 - 20x + x^2 = 58 && \{\text{expand the brackets}\} \\ \therefore \quad &2x^2 - 20x + 100 - 58 = 0 && \{\text{subtract 58 from both sides}\} \\ \therefore \quad &2x^2 - 20x + 42 = 0 && \{\text{collect like terms}\} \\ \therefore \quad &x^2 - 10x + 21 = 0 && \{\text{divide each term by the common factor 2}\} \\ \therefore \quad &(x-3)(x-7) = 0 && \{\text{factorise}\} \\ \therefore \quad &x = 3 \text{ or } 7 \end{aligned}$$

If $x = 3$, $10 - x = 7$.
If $x = 7$, $10 - x = 3$.

$\therefore$ the numbers are 3 and 7.

EXERCISE 19E

1. The sum of a number and its square is 42. Find the number.
2. When a number is squared, the result is five times the original number. Find the number.
3. When a number is subtracted from its square, the result is 56. Find the number.
4. Find two consecutive integers such that the square of the smaller plus twice the larger is 50.
5. Two numbers have a sum of 9 and the sum of their squares is 153. Find the numbers.
6. The product of two consecutive integers is 156. Find the integers.
7. The sum of the squares of two consecutive odd numbers is 290. Find the numbers.
8. When the square of a number is added to the number trebled the result is 108. What is the number?
9. Two numbers differ by 4. The product of the two numbers is 221. What are the numbers?

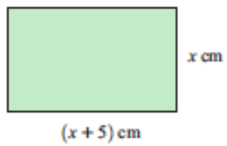
Example 7 Self Tutor

A rectangle has length 3 cm greater than its width. If it has an area of 28 cm^2, find the dimensions of the rectangle.

If x cm is the width, then $(x+3)$ cm is the length.

$$\begin{aligned} \therefore \quad x(x+3) &= 28 && \{\text{width} \times \text{length} = \text{area}\} \\ \therefore \quad x^2 + 3x &= 28 && \{\text{expanding}\} \\ \therefore \quad x^2 + 3x - 28 &= 0 && \{\text{writing with RHS} = 0\} \\ \therefore \quad (x+7)(x-4) &= 0 && \{\text{factorising LHS}\} \\ \therefore \quad x + 7 = 0 \quad &\text{or} \quad x - 4 = 0 && \{\text{Null Factor law}\} \\ \therefore \quad x &= -7 \text{ or } 4 \\ \therefore \quad x &= 4 && \{\text{lengths must be positive}\} \end{aligned}$$

$\therefore$ rectangle is 4 cm $\times$ 7 cm.

10. The length of a rectangle is 4 cm more than the width. If the area is 96 cm^2, find the width of the rectangle.
11. 

   a. Write down an expression for the rectangle's:
      i. area ii. perimeter.
   b. If the perimeter is 21.6 cm, what is the rectangle's area?
   c. If the area is 176 cm^2, what is the rectangle's perimeter?
12. A triangle has altitude 4 cm less than its base. If the area is $38\frac{1}{2}$ cm^2, find the base of the triangle.
13. A rectangle has sides which differ in length by 3 cm. If the area is 154 cm^2, find the perimeter of the rectangle.

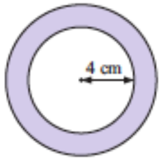
14 A rectangle has length 4 cm longer than its width. It has an area numerically equal to its perimeter.

a Find the width of the rectangle. **b** Check your answer to **a**.

15 A floor is 18 m by 16 m. A carpet is placed in its centre so that an equal width of uncovered floor remains all the way around the carpet. The area of the floor remaining is 120 m^2. Find the dimensions of the carpet.

Hint: Let the remaining strip of floor have width x m.

16



The shaded region has area 33π cm^2.
Find the radius of the outer circle.

17 **a** Kelly claims that the equation $x^2 + ax + b = 0$ has solutions $x = 2$ and $x = -5$ but has forgotten the values of a and b. Find a and b to help Kelly.

b Fredrik knows that $x^2+ax+b=0$ has only one solution and this is $x = -1\frac{1}{2}$. What are the values of a and b?

18 The sum of the squares of three consecutive integers is 110.

a Let the smallest integer be x and find a quadratic equation involving x.

b Solve the quadratic equation and hence write down the three integers.

c This time let the middle integer be x and find the quadratic equation involving x.

d Solve the equation in **c** and hence write down the three integers.

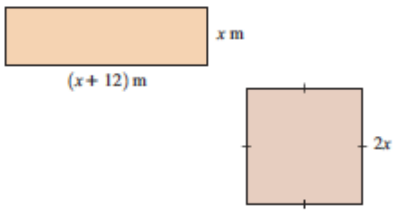
e Which method is better, starting with the smallest being x or the middle number being x? Why?

19 The sum of the squares of three consecutive odd numbers is 371. What are the numbers?

20 Three numbers are of the form $x-1$, x and $x+1$.
The square of the middle number is twice the product of the other two numbers.
What are the numbers given that each of them is positive?

21 The sum of the squares of five consecutive integers is 15. What are the integers?

22



Alongside are the dimensions of two flower beds of equal area.

a Find their dimensions.

b What is the total length of border needed to enclose both of them?

F SIMULTANEOUS EQUATIONS INVOLVING QUADRATIC EQUATIONS

Often the solution of simultaneous equations involves a quadratic equation. To solve these problems we use **substitution** of one equation into the other.

Example 8 Self Tutor

Solve simultaneously: $y = x^2$ and $y = 2x + 3$.

We substitute x^2 for y in the second equation.

$$\begin{aligned} \text{So, } x^2 &= 2x+3 \\ \therefore\ x^2 - 2x - 3 &= 0 \\ \therefore\ (x-3)(x+1) &= 0 \\ \therefore\ x-3 = 0 \ \text{ or } \ x+1 &= 0 \\ \therefore\ x = 3 \ \text{ or } \ x &= -1 \end{aligned}$$

When $x = 3$, $y = 3^2 = 9$.
When $x = -1$, $y = (-1)^2 = 1$.

Thus the solutions are $x = 3$, $y = 9$ or $x = -1$, $y = 1$.

Example 9 Self Tutor

Solve simultaneously: $xy = -6$ and $y = x - 5$

We substitute $(x-5)$ for y in the first equation.

$$\begin{aligned} \text{So, } x(x-5) &= -6 \\ \therefore\ x^2 - 5x + 6 &= 0 \\ \therefore\ (x-2)(x-3) &= 0 \\ \therefore\ x-2 = 0 \ \text{ or } \ x-3 &= 0 \\ \therefore\ x = 2 \ \text{ or } \ x &= 3 \end{aligned}$$

When $x = 2$, $y = 2 - 5 = -3$.
When $x = 3$, $y = 3 - 5 = -2$.

Thus the solutions are $x = 2$, $y = -3$ or $x = 3$, $y = -2$.

EXERCISE 19F

1 Solve simultaneously:

a $y = x + 6$ and $y = x^2$

b $y = x + 12$ and $y = x^2$

c $y = x^2$ and $y = 2x - 1$

d $y = x^2$ and $x + y = 12$

e $y = 8 - x$ and $xy = 7$ **f** $xy = -5$ and $y = x - 6$

g $xy = 4$ and $y = 4 - x$ **h** $xy + 6 = 0$ and $x + y = 1$

2 Two numbers have a product of 24 and one of them is five more than the other. What are the numbers?

3 One number is the square of another number and is also 8 more than twice that number. What are the numbers?

LINKS click here

HOW FAR WILL A CAR TRAVEL WHEN BRAKING?

Areas of interaction:
Community and service, Health and social education

REVIEW SET 19A

1 Solve for x:

a $-5x = 0$ **b** $2(x+2)(x-8) = 0$ **c** $-2(2x-1)^2 = 0$

d $4x^2 + 8x = 0$ **e** $x^2 - 5x - 24 = 0$ **f** $x^2 = 7x + 18$

2 Solve for x:

a $x^2 - 9 = x + 3$ **b** $x(x+4) = 8x - 3$

3 Solve simultaneously:

a $y = x^2$ and $y = x + 2$ **b** $x^2 + y^2 = 25$ and $y = x + 7$.

4 If a number is subtracted from its square, the result is 56. What is the number?

5 A rectangle has length 6 cm larger than its width and has area 72 cm^2. What is the width of the rectangle?

6 The height of a triangle is 4 cm more than its base. Find the length of its base given that its area is 58.5 cm^2.

7 A square has sides of length $3x$ cm. A rectangle is $2x$ cm by $(x+7)$ cm. The area of the square is twice that of the rectangle. Find the dimensions of each figure.

8 Two numbers have a product of -6 and one of them is 5 more than the other. Find the numbers.

REVIEW SET 19B

1 Solve for x:

a $-4x = 0$ **b** $3x(2x-1) = 0$ **c** $x^2 = 14x - 33$

d $(1-3x)^2 = 0$ **e** $5x - 10x^2 = 0$ **f** $x^2 + 7x + 12 = 0$

2 Solve for x:

a $x(x-3) = 10$ **b** $(x+2)(x-3) = 2x(x-3) - 2$

3 Solve simultaneously:

a $xy = -8$ and $y = 2 - x$ **b** $x^2 + y = 10$ and $y = 2x - 5$.

4 The square of a number is subtracted from the original number and the result is -110. Find the number.

5 A rectangular plot has length 5 m more than its width. If its area is 84 m^2, find the dimensions of the plot.

6 A rectangle has one side 4 m longer and the other side 2 m shorter than a square. The rectangle's area is 14 m^2 more than that of the square. What is the area of the rectangle?

7 The sum of the squares of three consecutive even numbers is 308. What are the integers?

8 The sum of the squares of two numbers is 5 and one of the numbers is 3 more than the other. Find the numbers.

ADA BYRON LOVELACE
(1815 - 1852)

Ada Byron Lovelace was born in England. She was the daughter of the famous poet, **Lord Byron**. However, most of her basic mathematical knowledge was taught to her by her mother who was very interested and capable in the subject.

When friends of her parents recognised her ability they suggested that she should go and see **Charles Babbage** who was one of the notable mathematicians of that era. At that time Babbage was using inventions of his own which were actually very basic mechanical computers. He called two of these machines the *Difference Engine* and the *Analytic Engine*.

At the age of 18, Lovelace had easily grasped the concepts involved in these machines and was greatly motivated to pursue a study of mathematics. Babbage was so impressed by her insight into the workings of his equipment that he readily agreed when she asked if she could work with him.

Later she published several essays explaining how and why the machines worked. Simultaneously, she developed a language for "communicating with machines". This language served the same purpose as does present day computer programming.

Like most people, Ada enjoyed mathematical games, and she was delighted when she discovered winning strategies. Soon after Ada purchased the game of "Solitaire" from a toy shop she was able to complete it successfully every time. Not satisfied with this achievement she asked Babbage if it were possible to find a formula for achieving a successful solution to this game.